

The Quantum Eigenvalue Problem and Z-Eigenvalues of Tensors

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Abstract. The quantum eigenvalue problem arises in the study of the geometric measure of the quantum entanglement. In this paper, we convert the quantum eigenvalue problem to the Z-eigenvalue problem of a real symmetric tensor. In this way, the theory and algorithms for Z-eigenvalues can be applied to the quantum eigenvalue problem. In particular, this gives an upper bound for the number of quantum eigenvalues. We show that the quantum eigenvalues appear in pairs, i.e., if a real number λ is a quantum eigenvalue of a square symmetric tensor Ψ , then $-\lambda$ is also a quantum eigenvalue of Ψ . When Ψ is real, we show that the entanglement eigenvalue of Ψ is always greater than or equal to the Z-spectral radius of Ψ , and that in several cases the equality holds. We also show that the ratio between the entanglement eigenvalue and the Z-spectral radius of a real symmetric tensor is bounded above in a real symmetric tensor space of fixed order and dimension.

Key Words. quantum eigenvalue, Z-eigenvalue, real symmetric tensor, entanglement eigenvalue.

1 Introduction

Eigenvalues of higher order tensors were introduced in 2005 [27, 22] and have attracted much attention in the literature. The E-eigenvalues and E-eigenvectors of tensors were introduced in

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[27, 29]. For a real tensor, if an E-eigenvalue has a real E-eigenvector, then it is also real and called a Z-eigenvalue of that tensor. The corresponding E-eigenvector is called a Z-eigenvector. Z-eigenvalues found applications in determining positive definiteness of a multivariate system [27, 32], best rank-one approximation [27, 30, 32, 37], magnetic resonance imaging [5, 33], spectral hypergraph theory [16, 21] and symmetric embedding [34], and possess links with geometry [1, 2, 3, 4, 28]. Properties of and algorithms for finding Z-eigenvalues and Z-eigenvectors can be found in [8, 9, 10, 19, 32, 33, 36]. Z-eigenvalues are roots of E-characteristic polynomials, which have been studied in [7, 20, 23, 27, 29].

Theoretically, people were not very sure if Z-eigenvalues or E-eigenvalues are more essential. Actually, the definition of E-eigenvalues missed some eigenvalues in the matrix case [19, 29]. Another concept, equivalence eigenpairs were introduced to remedy this defect in [7]. People were also not sure if the normalizing constraint in the definition of Z-eigenvalues has any application background.

The quantum eigenvalue problem arises in the study of the geometric measure of the quantum entanglement [35]. The quantum entanglement problem is a central problem in quantum information [24]. The geometric measure of entanglement of a multipartite pure state, based upon the minimum distance of that state from the set of separable pure states, was introduced in [35] and has been studied in [6, 11, 18, 15, 14, 26, 31]. The quantum eigenvalues are real numbers. The largest quantum eigenvalue is called the **entanglement eigenvalue**, which is corresponding to the nearest separable pure state to the given pure state, and thus has its physical meaning.

Only recently, it was shown in [17] that the entanglement eigenvalue of a symmetric pure state with nonnegative amplitudes is equal to the largest Z-eigenvalue of the underlying nonnegative tensor.

In this paper, we convert the quantum eigenvalue problem to a Z-eigenvalue problem of a real symmetric tensor. In this way, on one hand, the theory and algorithms for Z-eigenvalues can be applied to the quantum eigenvalue problem. For example, we now know that the number of quantum eigenvalues of a multipartite pure state is finite, and we know an upper bound of that number. These were previously unknown. On the other hand, we now know Z-eigenvalues are more essential, E-eigenvalues and equivalence eigenpairs are some useful concepts for studying Z-eigenvalues, and the normalizing constraint for Z-eigenvalues has some physics background in this case.

We show that the quantum eigenvalues appear in pairs, i.e., if a real number λ is a quantum eigenvalue of a square symmetric tensor Ψ , then $-\lambda$ is also a quantum eigenvalue of Ψ . When Ψ is real, we show that the entanglement eigenvalue of Ψ is always greater than or equal to the Z-spectral radius of Ψ , and that in several cases the equality holds.

The rest of the paper is organized in the following way. In the next section, the Z-eigenvalue problem, the quantum eigenvalue problem and some related concepts are reviewed. We show there that the quantum eigenvalues appear in pairs. In Section 3, we show that when the underlying symmetric tensor Ψ is real, the entanglement eigenvalue of Ψ is always greater than or equal to the Z-spectral radius of Ψ , and that in several cases the equality holds. In general, the equality may not hold. Then, for the m -order n -dimensional real symmetric tensor space, we introduce the QR-ratio which is the maximum ratio between the entanglement eigenvalues and the Z-spectral radii of the tensors in that space. We convert the quantum eigenvalue problem to the Z-eigenvalue problem of a real symmetric tensor, and discuss the consequences of this result in Section 4. We also show in Section 4 that the QR-ratio is finite there. Some final remarks are made in Section 5.

Throughout this paper, we assume that $m, n \geq 2$.

2 The Z-eigenvalue problem and the quantum eigenvalue problem

Suppose that $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_m})$ is an m -order n -dimensional tensor, where $\mathcal{T}_{i_1 \dots i_m} \in \mathbb{C}$ for $i_1, \dots, i_m = 1, \dots, n$. If there are a number $\lambda \in \mathbb{C}$ and a nonzero vector $w \in \mathbb{C}^n$ such that

$$\begin{cases} \mathcal{T}w^{m-1} &= \lambda w, \\ w^\top w &= 1, \end{cases} \quad (1)$$

where $\mathcal{T}w^{m-1} \in \mathbb{C}^n$ with its i th component defined by

$$(\mathcal{T}w^{m-1})_i = \sum_{i_2, \dots, i_m=1}^m \mathcal{T}_{ii_2 \dots i_m} w_{i_2} \cdots w_{i_m},$$

then we call λ an **E-eigenvalue** of \mathcal{T} with w as its corresponding **E-eigenvector**. We always have

$$\lambda = \mathcal{T}w^m \equiv \sum_{i_1, \dots, i_m=1}^m \mathcal{T}_{i_1 \dots i_m} w_{i_1} \cdots w_{i_m}.$$

If \mathcal{T} and w are real, then λ is also real. In this case, we call λ a **Z-eigenvalue** of \mathcal{T} with w as its corresponding **Z-eigenvector**. By [27], if \mathcal{T} is real and symmetric, Z-eigenvalues always exist. In [30], the largest absolute value of Z-eigenvalues of \mathcal{T} is called the spectral radius of \mathcal{T} . In [9], this value is called the **Z-spectral radius** of \mathcal{T} . Since there are other kinds of eigenvalues of tensors, we use the name Z-spectral radius in this paper. We denote the Z-spectral radius of \mathcal{T} as $Z(\mathcal{T})$. For more properties of Z-eigenvalues, we refer to the references cited in the first paragraph of this paper.

Define the Frobenius norm of \mathcal{T} as

$$\|\mathcal{T}\| = \sqrt{\sum_{i_1, \dots, i_m=1}^n \mathcal{T}_{i_1 \dots i_m}^2}.$$

We say that tensor \mathcal{T} is **symmetric** if its entries $\mathcal{T}_{i_1 \dots i_m}$ are invariant under any permutation of its indices.

Denote the space of m -order n -dimensional real symmetric tensors by $S(m, n)$. Denote the zero tensor in $S(m, n)$ by \mathcal{O} . It was proved in [30] that the Z-spectral radius is also a norm of $S(m, n)$. By the norm equivalence theorem in finite dimensional spaces [25], both

$$\rho(m, n) := \sup \left\{ \frac{Z(\mathcal{T})}{\|\mathcal{T}\|} : \mathcal{T} \in S(m, n) \setminus \{\mathcal{O}\} \right\}$$

and

$$\mu(m, n) := \inf \left\{ \frac{Z(\mathcal{T})}{\|\mathcal{T}\|} : \mathcal{T} \in S(m, n) \setminus \{\mathcal{O}\} \right\}$$

are finite positive numbers. In [30], $\mu(m, n)$ is called the **best rank-one approximation ratio** of $S(m, n)$.

In this paper, we only discuss the quantum eigenvalue problem arising from the geometric measure of entanglement of a multipartite symmetric pure state. The quantum eigenvalue problem for the geometric measure of entanglement of a multipartite nonsymmetric pure state can be converted to the symmetric one by symmetric embedding [34], see [17].

An m -partite symmetric pure state $|\Psi\rangle$ can be regarded as an element in a tensor product space $\mathcal{H}^{\otimes m}$, satisfying $\langle \Psi | \Psi \rangle = 1$, where the dimension of \mathcal{H} is n . An m -partite symmetric pure separable state has the form $|\phi\rangle^{\otimes m}$, where $|\phi\rangle \in \mathcal{H}$ and $\langle \phi | \phi \rangle = 1$. Then, by [35, 18, 31], the

quantum eigenvalue problem has the form

$$\begin{cases} \langle \Psi | \phi \rangle^{\otimes(m-1)} = \lambda \langle \phi |, \\ \langle \phi^{\otimes(m-1)} | \Psi \rangle = \lambda | \phi \rangle, \\ \langle \phi | \phi \rangle = 1, \\ \langle \Psi | \Psi \rangle = 1. \end{cases} \quad (2)$$

We assume that under an orthonormal basis of \mathcal{H} , by abusing the notation, the state $|\Psi\rangle$ is represented by an m -order n -dimensional symmetric complex tensor $\Psi = (\Psi_{i_1 \dots i_m})$, and the state $|\phi\rangle$ is represented by a vector $z \in \mathbb{C}^n$. The assumption that $\langle \phi | \phi \rangle = 1$ implies that $\bar{z}^\top z = 1$, where the bar denotes the conjugate. We keep this assumption. The assumption that $\langle \Psi | \Psi \rangle = 1$ implies that the Frobenius norm of Ψ should be one. We drop this assumption to accommodate more general problems. Throughout this paper, we assume that Ψ is symmetric.

With above notation, the quantum eigenvalue problem (2) has the following form.

$$\begin{cases} \Psi z^{m-1} = \lambda \bar{z}, \\ \bar{\Psi} \bar{z}^{m-1} = \lambda z, \\ \bar{z}^\top z = 1. \end{cases} \quad (3)$$

Suppose $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$ satisfy (3). Then λ must be real. We call it a **quantum eigenvalue** or in short a **Q-eigenvalue** of the tensor Ψ , and call z a **Q-eigenvector** of Ψ , associated with the Q-eigenvalue λ . The largest Q-eigenvalue of Ψ is called the **entanglement eigenvalue** of Ψ , and denoted as $Q(\Psi)$. It is associated with the nearest separable pure state of the state $|\Psi\rangle$.

In fact, (3) is equivalent to

$$\begin{cases} \Psi z^{m-1} = \lambda \bar{z}, \\ \bar{z}^\top z = 1, \\ \lambda \in \mathbb{R}. \end{cases} \quad (4)$$

By [31, 35], Q-eigenvalues always exist.

Because of the conjugate operation is involved in (3) and (4), the theory of algebraic geometry [12, 13] cannot be applied to them directly. This poses difficulty to analyze them. Before this paper, it was unknown even if there are a finite number of Q-eigenvalues or not.

We now prove a property of Q-eigenvalues.

Proposition 2.1 *The Q-eigenvalues appear in pairs, i.e., if a real number λ is a Q-eigenvalue of a square symmetric tensor Ψ , then $-\lambda$ is also a Q-eigenvalue of Ψ .*

Proof. Suppose that λ is a Q-eigenvalue of Ψ with its corresponding Q-eigenvector z . Then $-\lambda$ is a Q-eigenvalue of Ψ with its corresponding Q-eigenvector $ze^{\frac{\pi\sqrt{-1}}{m}}$. The conclusion follows. \square

3 The case that Ψ is real

Recently, it was proven in [17] that if furthermore Ψ is nonnegative, then $Q(\Psi) = Z(\Psi)$. We now establish results in more general cases.

Theorem 3.1 *If Ψ is real, then $Q(\Psi) \geq Z(\Psi)$. In the following six cases, we have $Q(\Psi) = Z(\Psi)$. We assume that Ψ is real in all these six cases.*

- 1). $m = 2$;
- 2). Ψ is diagonal;
- 3). Ψ is nonnegative;
- 4). Ψ is nonpositive;
- 5). $\Psi = \sum_{k=1}^n \alpha_k (y^{(k)})^m$ where $\alpha_k, k = 1, \dots, n$, are real numbers, $\{y^{(k)} : k = 1, \dots, n\}$ is an orthonormal basis of \mathbb{R}^n ;
- 6). $m \geq 4$ is even, an element of Ψ is nonzero only if a half of its indices are the same and the other half of its indices are also the same, and Ψ is diagonally dominated in the sense that the absolute value of each of its diagonal elements is greater than or equal to $\left(\frac{m-1}{2}\right)$ times of the absolute value of any off-diagonal element whose indices overlap with that diagonal element.

Proof. By the definitions of Q-eigenvalues and Z-eigenvalues, we see that if Ψ is real, each Z-eigenvalue of Ψ is also a Q-eigenvalue of Ψ . By Proposition 2.1, if λ is a Q-eigenvalue of Ψ , then $-\lambda$ is also a Q-eigenvalue of Ψ . Putting these together, we have $Q(\Psi) \geq Z(\Psi)$ when Ψ is real.

- 1). This case can be regarded as a special case of case 5). Here we give a direct proof for this case. Assume that λ is the entanglement eigenvalue of Ψ with z as its corresponding Q-eigenvector. Assume that $z = x + y\sqrt{-1}$, where x and y are real. Then we have $\Psi x = \lambda x$, $\Psi y = -\lambda y$ and at least one of x and y is not equal to zero. If $x \neq 0$, then λ is a Z-eigenvalue of Ψ with $\frac{x}{\sqrt{x^2 + y^2}}$ as its corresponding Z-eigenvector. If $y \neq 0$, then $-\lambda$ is a Z-eigenvalue of Ψ with $\frac{y}{\sqrt{x^2 + y^2}}$ as its corresponding Z-eigenvector.

$\frac{y}{\sqrt{y^\top y}}$ as its corresponding Z-eigenvector. This shows that $Q(\Psi) \leq Z(\Psi)$. Since we always have $Q(\Psi) \geq Z(\Psi)$, we have $Q(\Psi) = Z(\Psi)$.

2). Suppose that Ψ is real and diagonal. Denote the diagonal elements of Ψ by $a_{k\dots k}$ for $k = 1, \dots, n$. Assume that $|a_{i\dots i}| = \max\{|a_{k\dots k}| : k = 1, \dots, n\}$. Let $x_i = 1$ and $x_k = 0$ for $k \neq i$. Let $\mu = a_{i\dots i}$. Then μ is a Z-eigenvalue of Ψ with x as its corresponding Z-eigenvector. This shows that $Z(\Psi) \geq |a_{i\dots i}|$. Let λ be the entanglement eigenvalue of Ψ with z as its corresponding Q-eigenvector. Assume that $z_j \neq 0$. Then $|z_j| \leq \sqrt{\bar{z}^\top z} = 1$, and we have $a_{j\dots j} z_j^{m-1} = \lambda \bar{z}_j$, which implies that

$$Q(\Psi) = \lambda \leq |a_{j\dots j}| \cdot |z_j|^{m-2} \leq |a_{i\dots i}| \leq Z(\Psi).$$

Since we always have $Q(\Psi) \geq Z(\Psi)$, we have $Q(\Psi) = Z(\Psi)$.

3). This was proved in [17].

4). By definition, $Z(\Psi) = Z(-\Psi)$. By Proposition 2.1, $Q(\Psi) = Q(-\Psi)$. By 3), $Q(-\Psi) = Z(-\Psi)$. The conclusion follows now.

5). It suffices to prove that in this case, if λ is a Q-eigenvalue of Ψ , then either λ or $-\lambda$ is a Z-eigenvalue of Ψ . Suppose that λ is a Q-eigenvalue of Ψ , with Q-eigenvector z . Then $\Psi z^{m-1} = \lambda \bar{z}$. That is,

$$\sum_{i=1}^n \alpha_k (y^{(k)})^\top z)^{m-1} y^{(k)} = \lambda \bar{z}.$$

For convenience of notation, we denote $\beta_k = y^{(k)\top} z = r_k e^{\theta_k \sqrt{-1}}$ for $k = 1, 2, \dots, n$. Since $y^{(1)}, \dots, y^{(n)}$ are orthonormal vectors, there exists k such that $\beta_k = y^{(k)\top} z = r_k e^{\theta_k \sqrt{-1}} \neq 0$.

Multiplying $(y^{(k)})^\top$ on the both sides of above equality, we have $\alpha_k r_k^{m-2} e^{m\theta_k \sqrt{-1}} = \lambda$, which implies that $m\theta_k = l\pi$ for an integer $l \geq 0$. Hence $\lambda = \alpha_k r_k^{m-2} \cos l\pi$ for all k satisfying $\beta_k \neq 0$.

Let $x = \sum_{j=1}^n r_j y^{(j)} \in \Re^n$. Then $x^\top x = 1$ and

$$\Psi x^{m-1} = \sum_{k=1}^n \alpha_k y^{(k)} r_k^{m-1} = \sum_{k=1}^n y^{(k)} (-1)^l \lambda r_k = (-1)^l \lambda x,$$

which means that $(-1)^l \lambda$ is a Z-eigenvalue of Ψ . The proof of 5) is completed.

6). Suppose that Ψ satisfies the conditions. Denote the elements of Ψ by $a_{k_1 \dots k_m}$ for $k_1, \dots, k_m = 1, \dots, n$. Assume that $|a_{i\dots i}| = \max\{|a_{k\dots k}| : k = 1, \dots, n\}$. Let $x_i = 1$ and $x_k = 0$ for $k \neq i$. Let $\mu = a_{i\dots i}$. Then μ is a Z-eigenvalue of Ψ with x as its corresponding

Z-eigenvector, as an element of Ψ is nonzero only if a half of its indices are the same and the other half of its indices are also the same. This shows that $Z(\Psi) \geq |a_{i\dots i}|$. Let λ be the entanglement eigenvalue of Ψ with z as its corresponding Q-eigenvector. Assume that $z_j \neq 0$. Then $|z_j| \leq \sqrt{\bar{z}^\top z} = 1$, and we have $a_{j\dots j} z_j^{m-1} + \sum_{k \neq j} \binom{m-1}{\frac{m}{2}} a_{j\dots j k \dots k} z_j^{\frac{m-2}{2}} z_k^{\frac{m}{2}} = \lambda \bar{z}_j$, which implies that

$$Q(\Psi) = \lambda \leq |a_{j\dots j}| \cdot \sum_{k=1}^n |z_k|^{\frac{m}{2}} \leq |a_{i\dots i}| \leq Z(\Psi).$$

Since we always have $Q(\Psi) \geq Z(\Psi)$, we have $Q(\Psi) = Z(\Psi)$. \square

In general, the equality $Q(\Psi) \geq Z(\Psi)$ may not hold. On Page 3 of [18], a counter example is given. By computation, for that example, we have $Z(\Psi) = 2.2805/\sqrt{21}$ while $Q(\Psi) = 3.1768/\sqrt{21}$.

If the equality $Q(\Psi) \geq Z(\Psi)$ holds, then the problem for finding the entanglement eigenvalue of a real symmetric tensor Ψ is turned out to the problem for finding the Z-spectral radius of Ψ . This reduces a half of the number of variables. Thus, a challenging task is to find a sufficient and necessary condition or some more general sufficient conditions such that this equality holds.

Another challenging task is to find the maximum ratio of $Q(\Psi)$ and $Z(\Psi)$. Define the **QR-ratio** of $S(m, n)$ as

$$R(m, n) = \sup \left\{ \frac{Q(\Psi)}{Z(\Psi)} : \Psi \in S(m, n) \setminus \{\mathcal{O}\} \right\}.$$

By Theorem 3.1, we have $R(2, n) = 1$ and $R(m, n) \geq 1$ for $m \geq 3$. In the next section, we will show that the QR-ratio $R(m, n)$ is a finite positive number. Even if we know that it is a finite positive number, what is its value, and is it attainable? These remain as topics for further research.

4 Converting the quantum eigenvalue problem to the Z-eigenvalue problem of a real symmetric tensor

In this section, we convert the quantum eigenvalue problem to the Z-eigenvalue problem of a real symmetric tensor.

Before proceeding, the following relation is established when $m = 2$.

Theorem 4.1 *Suppose that $m = 2$. Then there exists a symmetric matrix $M \in \mathbb{R}^{2n \times 2n}$ such that a number λ is a Q-eigenvalue of the tensor Ψ if and only if it is a Z-eigenvalue of the real*

symmetric matrix M . In this case, $z = x + y\sqrt{-1}$ is a Q -eigenvector of Ψ , if and only if $w = \begin{pmatrix} x \\ y \end{pmatrix}$ is a unit eigenvector of M , associated with λ .

Furthermore, the eigenvalues of M appear in pairs, i.e., if λ is an eigenvalue of M , then $-\lambda$ is also an eigenvalue of M .

Proof. For convenience of notation, let $\Psi = A + B\sqrt{-1}$ and $z = x + y\sqrt{-1}$ be a Q -eigenvector of Ψ associated with the Q -eigenvalue λ . Then $\Psi z = \lambda \bar{z}$ and $\bar{z}^\top z = 1$, that is

$$\begin{cases} Ax - By = \lambda x, \\ -Bx - Ay = \lambda y, \\ w^\top w = 1. \end{cases}$$

Let $M = \begin{pmatrix} A & -B \\ -B & -A \end{pmatrix}$ and $w = \begin{pmatrix} x \\ y \end{pmatrix}$. Then M is a real symmetric matrix, and the above system holds if and only if

$$\begin{cases} Mw = \lambda w, \\ w^\top w = 1. \end{cases}$$

Furthermore, assume that λ is an eigenvalue of M , with an eigenvector w . Denote $w = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\hat{w} = \begin{pmatrix} y \\ -x \end{pmatrix}$. Then we see that $-\lambda$ is also an eigenvalue of M with eigenvector \hat{w} . This completes our proof.

□

Note that since M is real symmetric, all of its eigenvalues are real, thus Q -eigenvalues of Ψ .

Now we are ready to extend the result to the m -partite case with $m \geq 3$. For $i = 1, \dots, 2n$, define $\hat{i} = i$ if $i \leq n$, and $\hat{i} = i - n$ if $i > n$. For an m -order n -dimensional tensor $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_m})$, two vectors $u = (u_i), v = (v_i) \in \mathbb{C}^n$ and $k = 0, \dots, m-1$, we have $\mathcal{T}u^{m-1-k}v^k \in \mathbb{C}^n$, defined by

$$\left(\mathcal{T}u^{m-1-k}v^k\right)_i = \sum_{i_2, \dots, i_m=1}^n \mathcal{T}_{i i_2 \dots i_m} u_{i_2} \dots u_{i_{m-k}} v_{i_{m-k+1}} \dots v_{i_m}.$$

Theorem 4.2 Suppose that Ψ is an m -order n -dimensional symmetric tensor, where $m \geq 3$.

Then there exists a real symmetric tensor $\mathcal{T} \in \mathbb{R}^{\overbrace{2n \times \dots \times 2n}^m}$ such that λ is a Q -eigenvalue of Ψ if and only if it is a Z -eigenvalue of \mathcal{T} . In this case, $z = x + y\sqrt{-1}$ is a Q -eigenvector of Ψ , if and only if $w = \begin{pmatrix} x \\ y \end{pmatrix}$ is a Z -eigenvector of \mathcal{T} , associated with λ .

Furthermore, the E -eigenvalues and Z -eigenvalues of \mathcal{T} appear in pairs, i.e., if λ is an E -eigenvalue or Z -eigenvalue of \mathcal{T} , then $-\lambda$ is also an E -eigenvalue or Z -eigenvalue of \mathcal{T} correspondingly.

Proof. We assume that $\Psi = \mathcal{A} + \mathcal{B}\sqrt{-1}$ is an m -order n -dimensional symmetric tensor, where tensors \mathcal{A} and \mathcal{B} are real symmetric tensors.

Suppose that $z = x + y\sqrt{-1}$ is a Q -eigenvector of Ψ associated with the Q -eigenvalue λ . Let $w = \begin{pmatrix} x \\ y \end{pmatrix}$. Then we have $\Psi z^{m-1} = \lambda \bar{z}$ and $\bar{z}^\top z = 1$, that is,

$$\begin{cases} (\mathcal{A} + \mathcal{B}\sqrt{-1}) \sum_{k=0}^{m-1} (\sqrt{-1})^k \binom{m-1}{k} x^{m-1-k} y^k = \lambda (x - y\sqrt{-1}), \\ w^\top w = 1. \end{cases}$$

Considering the two cases $k = 2j$ and $k = 2j + 1$, we have

$$\begin{cases} (\mathcal{A} + \mathcal{B}\sqrt{-1}) \left[\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^j \binom{m-1}{2j} x^{m-1-2j} y^{2j} + \sqrt{-1} \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^j \binom{m-1}{2j+1} x^{m-2j-2} y^{2j+1} \right] \\ = \lambda (x - y\sqrt{-1}), \\ w^\top w = 1. \end{cases}$$

Separating the real and imaginary parts, we have

$$\begin{cases} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^j \binom{m-1}{2j} \mathcal{A} x^{m-1-2j} y^{2j} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^{j+1} \binom{m-1}{2j+1} \mathcal{B} x^{m-2j-2} y^{2j+1} = \lambda x, \\ \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^{j+1} \binom{m-1}{2j+1} \mathcal{A} x^{m-2j-2} y^{2j+1} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{j+1} \binom{m-1}{2j} \mathcal{B} x^{m-1-2j} y^{2j} = \lambda y, \\ w^\top w = 1. \end{cases}$$

Let tensor $\mathcal{T} \in \mathbb{R}^{2n \times \dots \times 2n}$ be an m -order $2n$ -dimensional symmetric tensor, defined by

$$\mathcal{T}_{i_1 i_2 \dots i_m} = \begin{cases} (-1)^j \mathcal{A}_{i_1 \hat{i}_2 \dots \hat{i}_m} & \text{when there exist } 2j \text{ } (0 \leq j \leq \frac{m}{2}) \text{ indices larger than } n, \\ (-1)^{j+1} \mathcal{B}_{i_1 \hat{i}_2 \dots \hat{i}_m} & \text{when there exist } 2j+1 \text{ } (0 \leq j \leq \frac{m-1}{2}) \text{ indices larger than } n. \end{cases}$$

Then we have $\mathcal{T} w^{m-1} = \lambda w$. This indicates that λ is a Q -eigenvalue of tensor Ψ , associated with Q -eigenvector z if and only if λ is a Z -eigenvalue of tensor \mathcal{T} , associated with Z -eigenvector w .

Furthermore, assume that λ is an E -eigenvalue of \mathcal{T} , with an E -eigenvector $w = \begin{pmatrix} x \\ y \end{pmatrix}$. Here, x and y may not be real. Let $\hat{w} = \begin{pmatrix} y \\ -x \end{pmatrix}$. We may easily see that $-\lambda$ is also an E -eigenvalue of \mathcal{T} with E -eigenvector \hat{w} . Now assume that λ is a Z -eigenvalue of \mathcal{T} . In this case, we may assume

that x and y are real. Then both w and \hat{w} are real. Thus, in this case, $-\lambda$ is also a Z-eigenvalue of \mathcal{T} with Z-eigenvector \hat{w} . This completes our proof. \square

Remark 4.1 *The reformulated tensor \mathcal{T} is not unique. Assume that $m = 3$. For example, $w = \begin{pmatrix} x \\ y \end{pmatrix}$ is a Z-eigenvector of tensor \mathcal{T} , associated with the Z-eigenvalue λ , where \mathcal{T} is defined by*

$$\mathcal{T}_{ijk} = \begin{cases} \mathcal{A}_{ijk} & \text{if } i, j, k \leq n, \\ \mathcal{B}_{i\hat{j}\hat{k}} & \text{if one of } \{i, j, k\} > n, \text{ two of } \{i, j, k\} \leq n, \\ -\mathcal{A}_{i\hat{j}\hat{k}} & \text{if one of } \{i, j, k\} \leq n, \text{ two of } \{i, j, k\} > n, \\ -\mathcal{B}_{i\hat{j}\hat{k}} & \text{if } i, j, k > n, \end{cases}$$

when $z = x + y\sqrt{-1}$ is a Q-eigenvector of Ψ , associated with the Q-eigenvalue λ .

By Theorem 4.2 and the result in [7], we now have an upper bound for the number of Q-eigenvalues.

Proposition 4.1 *There are at most $\frac{(m-1)^{2n}-1}{m-2}$ Q-eigenvalues for a symmetric m -partite pure state $|\Psi\rangle$.*

By Theorem 4.2, we now also know Z-eigenvalues are more essential, E-eigenvalues and equivalence eigenpairs are some useful concepts for studying Z-eigenvalues, and the normalizing constraint for Z-eigenvalues has some physics background in this case.

We also note that \mathcal{T} is a special real symmetric tensor: its E-eigenvalues and Z-eigenvalues appear in pair. Do \mathcal{T} and its E-characteristic polynomial have other special properties? This leaves as a further research topic.

Finally, we prove that the QR-ratio $R(m, n)$ is finite.

Theorem 4.3 *The QR-ratio $R(m, n)$ is a finite positive number. Actually, we have*

$$1 \leq R(m, n) \leq \frac{2^{\frac{m-1}{2}} \rho(m, 2n)}{\mu(m, n)}.$$

Proof. The first inequality is by Theorem 3.1. Let $\Psi \in S(m, n)$. By Theorem 4.2, we have $Q(\Psi) = Z(\mathcal{T})$, where \mathcal{T} is a real symmetric tensor in $S(m, 2n)$, defined by

$$\mathcal{T}_{i_1 i_2 \dots i_m} = \begin{cases} (-1)^j \Psi_{i_1 \hat{i}_2 \dots \hat{i}_m} & \text{when there exist } 2j \text{ } (0 \leq j \leq \frac{m}{2}) \text{ indices larger than } n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\mathcal{T}\| = 2^{\frac{m-1}{2}}\|\Psi\|$. Thus, we have

$$Q(\Psi) = Z(\mathcal{T}) \leq \rho(m, 2n)\|\mathcal{T}\| = 2^{\frac{m-1}{2}}\rho(m, 2n)\|\Psi\| \leq \frac{2^{\frac{m-1}{2}}\rho(m, 2n)}{\mu(m, n)}Z(\Psi).$$

The conclusion follows. \square

As stated in the last section, this theorem initiates further study on the QR-ratio. what is its exact value? Is it attainable? These remain for future research.

5 Final remarks

This paper reveals that the quantum eigenvalue problem has a close relationship with the Z-eigenvalue problem. Two main results of this paper are Theorems 3.1 and 4.2. Theorem 3.1 discusses the possibility to convert the problem for finding the entanglement eigenvalue of a real symmetric tensor to the Z-spectral radius problem of the same tensor, while Theorem 4.2 converts the Q-eigenvalue problem of a complex symmetric tensor Ψ to the Z-eigenvalue problem of a real symmetric tensor \mathcal{T} , whose dimension is twice of the dimension of Ψ . The QR-ratio is introduced and proved to be finite. Further research on these is needed.

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